OpenGL® Lectures
Shearing Case Studies

By
Tom Duff
Pixar Animation Studios
Emeryville, California
and
George Ledin Jr
Sonoma State University
Rohnert Park, California

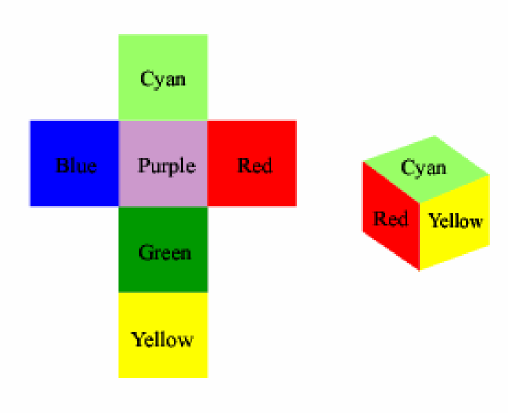
Case Study 1

Case Study Setup:

Assume in the world coordinates we have one color cube of size two, whose front is red.

colorcube4 ():

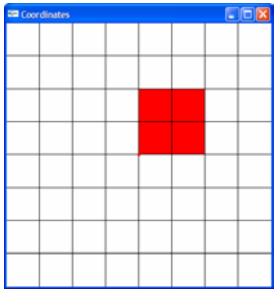
The front left right vertex is at (0,0,0)



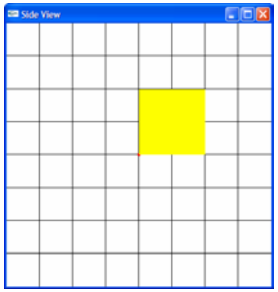
Goal:

We will observe how to do simple shearing with the cube without displacement using shearing matrix.

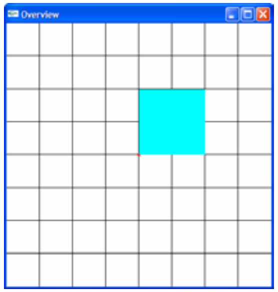
Front View



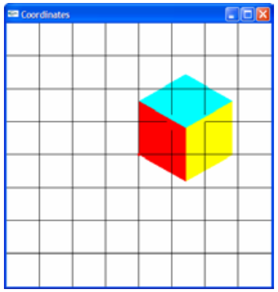
Side View



Top View



Diagonal View



Files Used:
Shearing.c, DrawCubes.c, DrawCubes.h, MyMatrix.c, MyMatrix.h

First, Set up the Projection View

```
glMatrixMode(GL_PROJECTION);
     glLoadIdentity();
     glOrtho(-4.0,4.0,-4.0,
            4.0,-4.0,4.0);
```

- **This set up means two things:**
  - If the object is outside its bounding box, it cannot be viewed. If part of an object is outside, that part cannot be viewed.
  - The camera is always in the geometric center of its bounding box. The camera cannot see anything outside its box.

XY direction shearing Matrix

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0.5 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

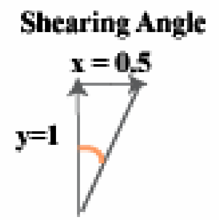
$$MxyShear = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & Shxy & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MxyShear will lean the object towards x direction.

Shxy:
  Y increase by 1 unit, X increase by Shxy unit.
  Shxy = tan(ShearingAngle)

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

X’ = X + Y*Shxy
Y’ = Y
Z’ = Z

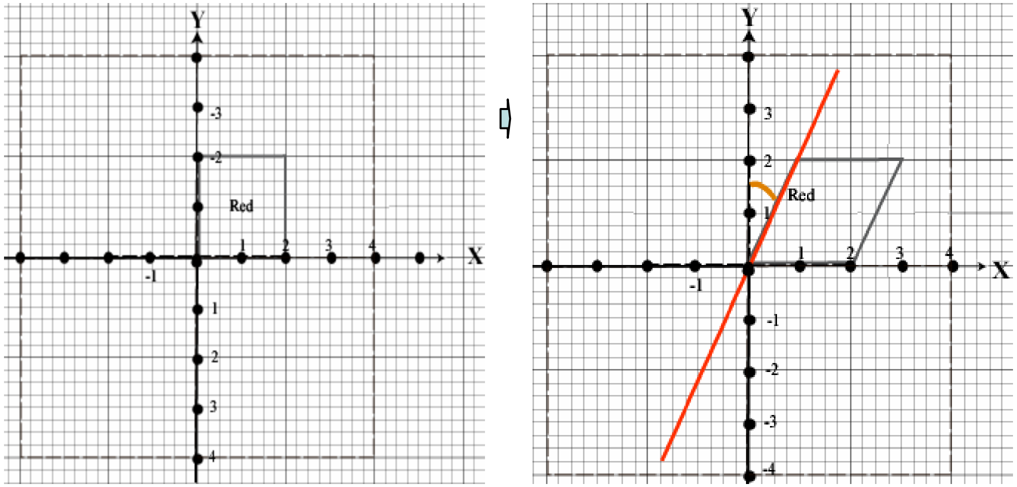


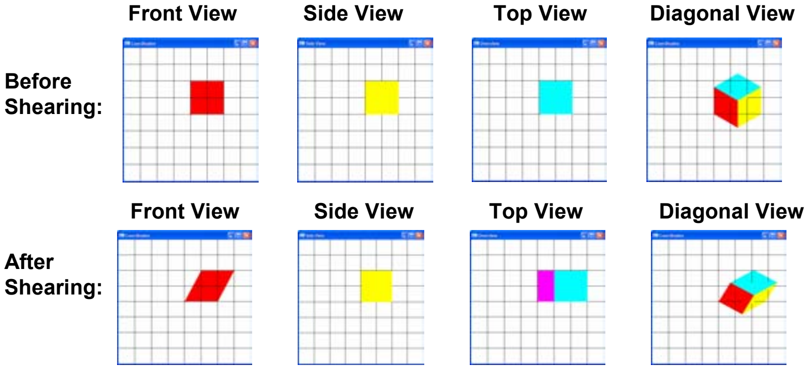
Example:

Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0,0,0)
P = (0,2,0) ➔ P’ = (1,2,0)

$$\left|\begin{array}{cccc} 1 & Shxy & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right| \overset{P}{\left|\begin{array}{c} X \\ Y \\ Z \\ 1 \end{array}\right|} = \left|\begin{array}{c} X + Y*Shxy \\ Y \\ Z \\ 1 \end{array}\right| = \overset{P'}{\left|\begin{array}{c} X' \\ Y' \\ Z' \\ 1 \end{array}\right|}$$

- **Case Study: Shxy = 0.5**





XZ direction shearing Matrix

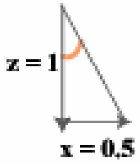
$$\text{MxzShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & Shxz & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MxzShear will shear the object in x direction.

Shxz:
  Z increase by 1 unit, X increase by Shxy unit.
  Shxz = tan(ShearingAngle)

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

Shearing Angle



X’ = X + Z*Shxz
Y’ = Y
Z’ = Z

Example:

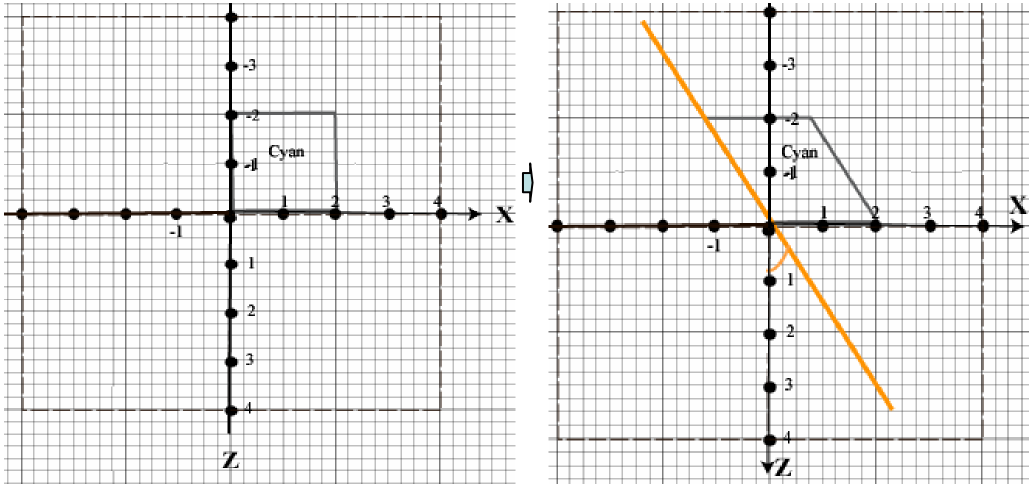
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0,0,0)
P = (0,2,0) ➔ P’ = (-1,2,0)

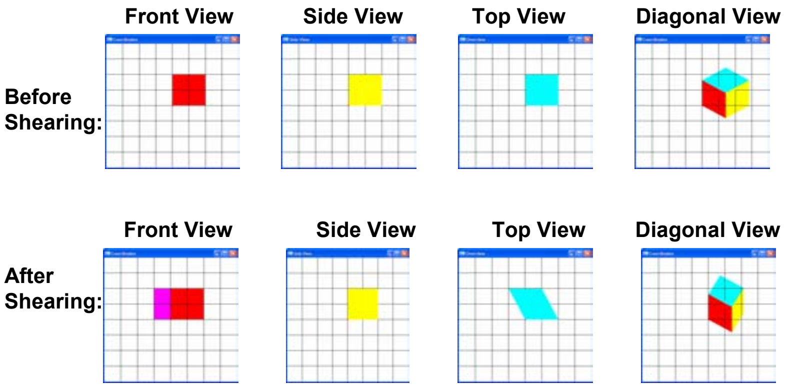
$$\begin{vmatrix} 1 & 0 & Shxz & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{P}{\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix}} = \begin{vmatrix} X + Z*Shxz \\ Y \\ Z \\ 1 \end{vmatrix} = \overset{P'}{\begin{vmatrix} X' \\ Y' \\ Z' \\ 1 \end{vmatrix}}$$

- **Case Study: Shxz = 0.5**

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0.5 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$





YX direction shearing Matrix

$$\mathbf{MyxShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & Shyx & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MyxShear will shear the object in y direction.

Shyx:
  X increase by 1 unit, Y increase by Shxy unit.
  Shxz = tan(ShearingAngle)

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

X’ = X
Y’ = Y + X*Shyx
Z’ = Z

Example:

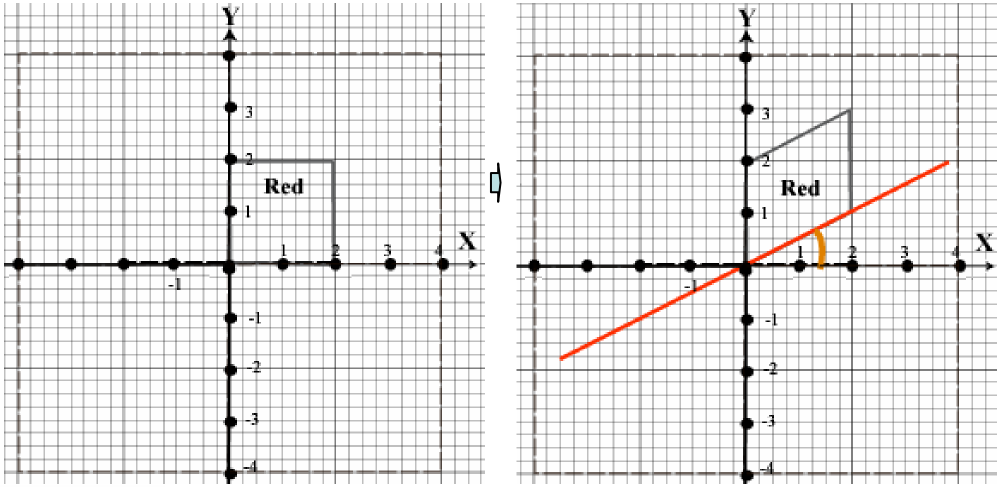
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0,0,0)
P = (2,0,0) ➔ P’ = (2,1,0)

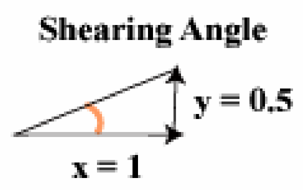
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ Shyx & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{P}{\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix}} = \begin{vmatrix} X \\ Y + X*Shyx \\ Z \\ 1 \end{vmatrix} = \overset{P'}{\begin{vmatrix} X' \\ Y' \\ Z' \\ 1 \end{vmatrix}}$$

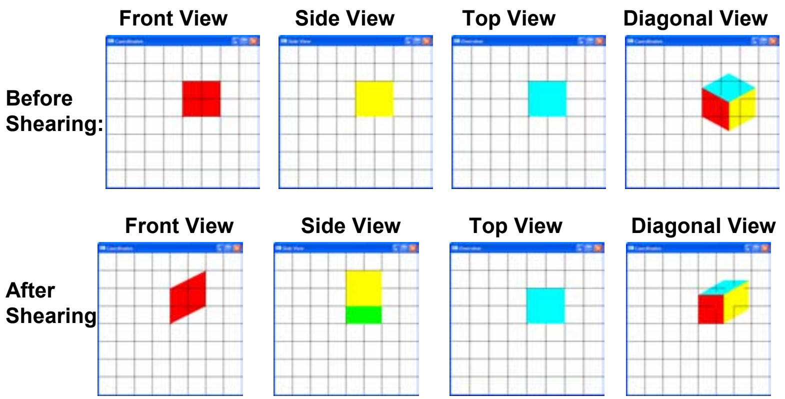
Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0.5 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

- **Example: Shyx = 0.5**







Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0.5 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

YZ direction shearing Matrix

$$MyzShear = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & Shyz & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MyzShear will shear the object in y direction.

Shyz:
Z increase by 1 unit, Y increase by Shxy unit.
Shxz = tan(ShearingAngle)

For every P = (Xz,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

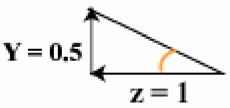
X’ = X
Y’ = Y + Z*Shyz
Z’ = Z

Example:

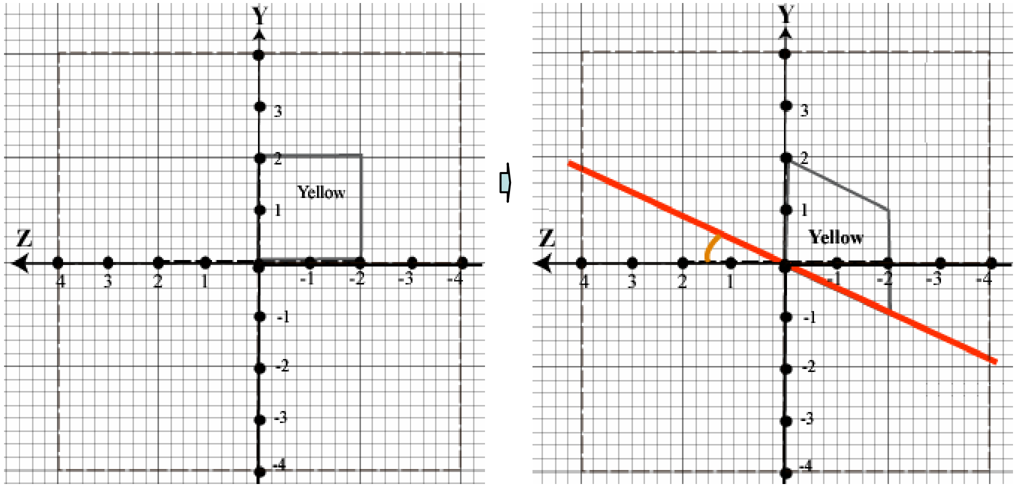
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0, 0,0)
P = (2,0,0) ➔ P’ = (2,-1,0)

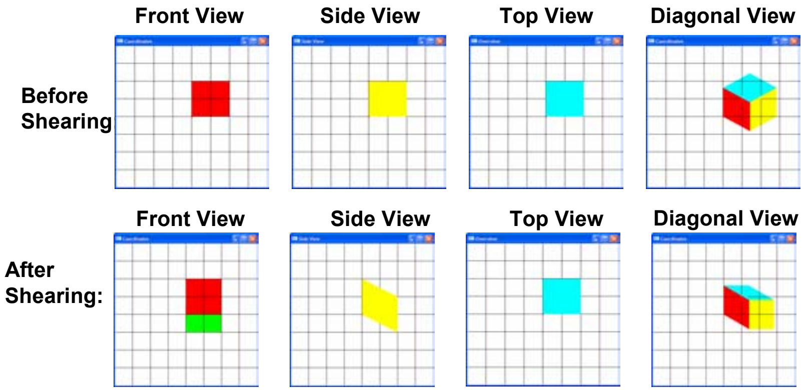
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & Shyz & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{P}{\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix}} = \begin{vmatrix} X \\ Y + Z*Shyz \\ Z \\ 1 \end{vmatrix} = \overset{P'}{\begin{vmatrix} X' \\ Y' \\ Z' \\ 1 \end{vmatrix}}$$

Shearing Angle



- Example: Shyz = 0.5





ZX direction shearing Matrix

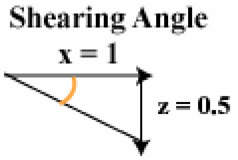
$$\text{MzxShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & \text{Shzx} & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MzxShear will shear the object in Z direction.

Shzx:
X increase by 1 unit, Z increase by Shxy unit.
Shxz = tan(ShearingAngle)

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

X’ = X
Y’ = Y
Z’ = Z + X*Shzx



Example:

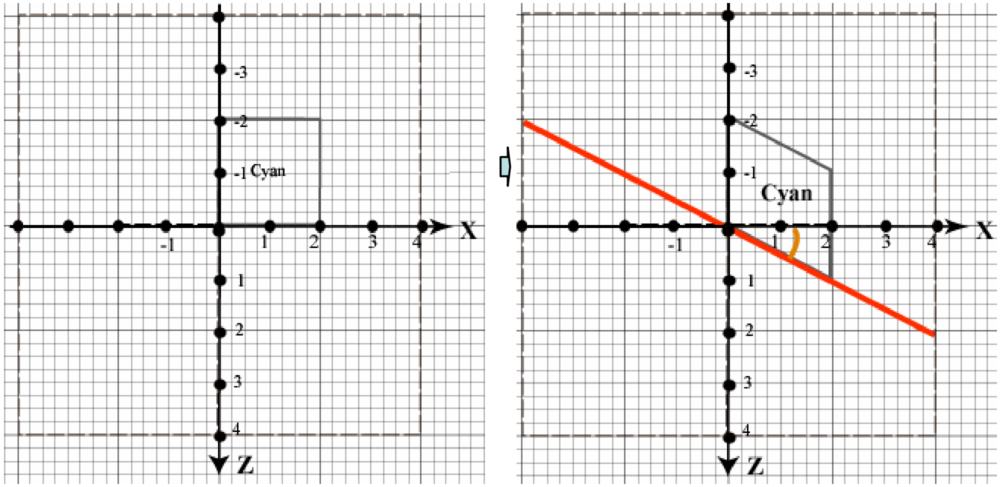
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0, 0, 0)
P = (2,0,0) ➔ P’ = (2, 0, -1)

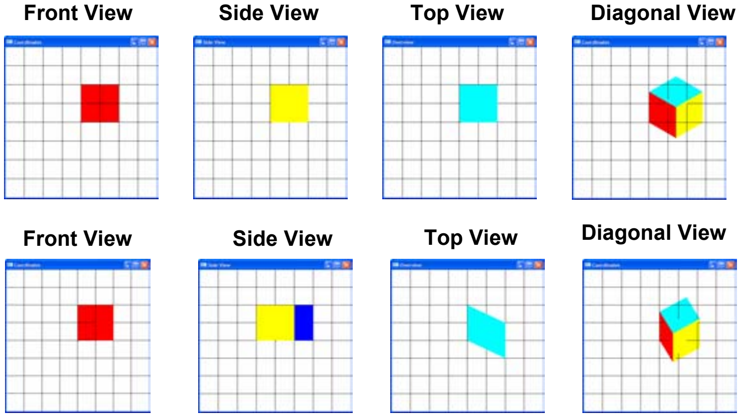
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \text{Shzx} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{P}{\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix}} = \begin{vmatrix} X \\ Y \\ Z + X*\text{Shzx} \\ 1 \end{vmatrix} = \overset{P'}{\begin{vmatrix} X' \\ Y' \\ Z' \\ 1 \end{vmatrix}}$$

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0.5 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

- **Example: Shzx = 0.5**




Front View **Side View** **Top View** **Diagonal View**

Before Shearing:

Front View **Side View** **Top View** **Diagonal View**

After Shearing:

ZY direction shearing Matrix

$$\text{MzyShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & Shzy & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MzyShear will shear the object in Z direction.

Shzy:

Y increase by 1 unit, Z increase by Shzy unit.

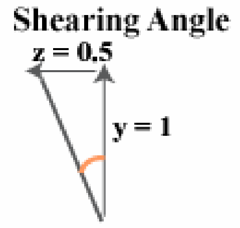
Shzy = tan(SearingAngle)

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

X’ = X

Y’ = Y

Z’ = Z + Y*Shzy



Example:

Shxy = tan(ShearingAngle) = 0.5

P = (0,0,0) ➔ P’ = (0, 0, 0)

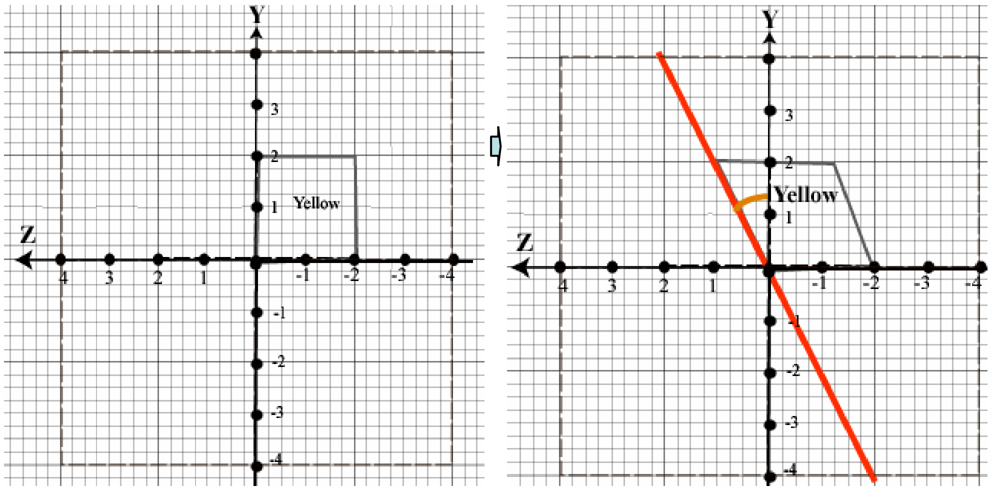
P = (0,2,0) ➔ p’ = (0, 2, 1)

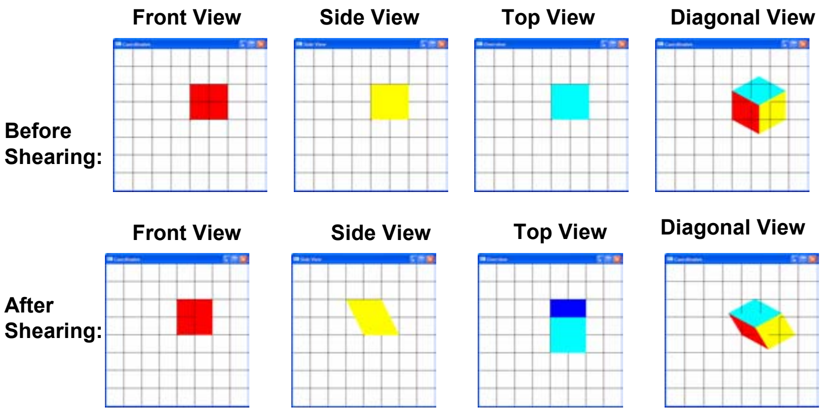
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & Shzy & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{P}{\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix}} = \begin{vmatrix} X \\ Y \\ Z + Y*Shzy \\ 1 \end{vmatrix} = \overset{P'}{\begin{vmatrix} X' \\ Y' \\ Z' \\ 1 \end{vmatrix}}$$

- **Example: Shzx = 0.5**

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0.5 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$





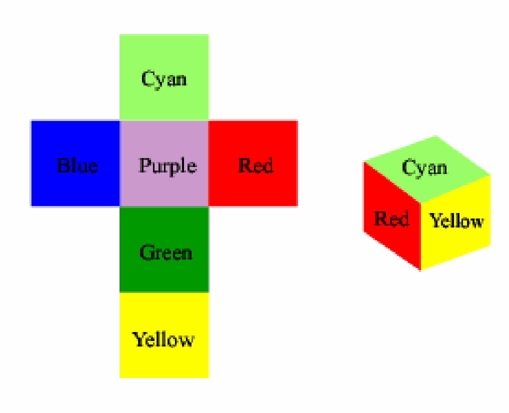
Case Study 2

Case Study Setup:

Assume in the world coordinates we have one color cube of size two, whose front is red.

colorcube4 ():

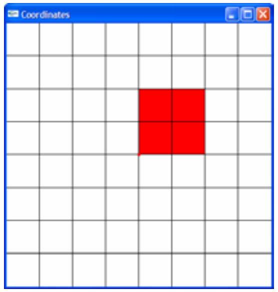
The front left right vertex is at (0,0,0)



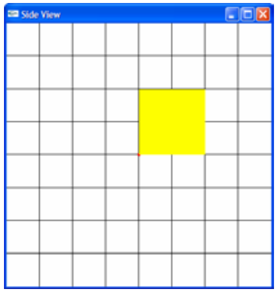
Goal:

We will observe how to shear and move the cube at the same time using shearing matrix.

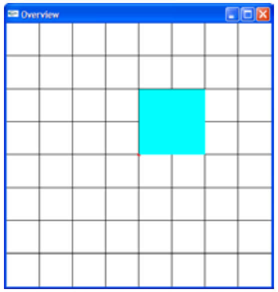
Front View



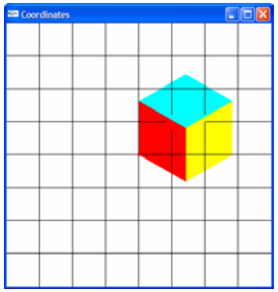
Side View



Top View



Diagonal View



Files Used:
Shearing.c, DrawCubes.c, DrawCubes.h, MyMatrix.c, MyMatrix.h

XY direction shearing Matrix with displacement

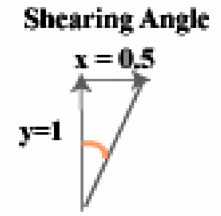
$$
\text{MxyShear} = \begin{array}{c|cccc|}
 & X & Y & Z & \\
X & 1 & Shxy & 0 & -Shxy*Yref \\
Y & 0 & 1 & 0 & 0 \\
Z & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 1
\end{array}
$$

MxyShear will lean the object towards x direction.

Shxy:
  Y increase by 1 unit, X increase by Shxy unit.
Yref:
  Shearing reference point on Y axis

For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

$X' = X + Shyx * (Y - Yref)$
$Y' = Y$
$Z' = Z$



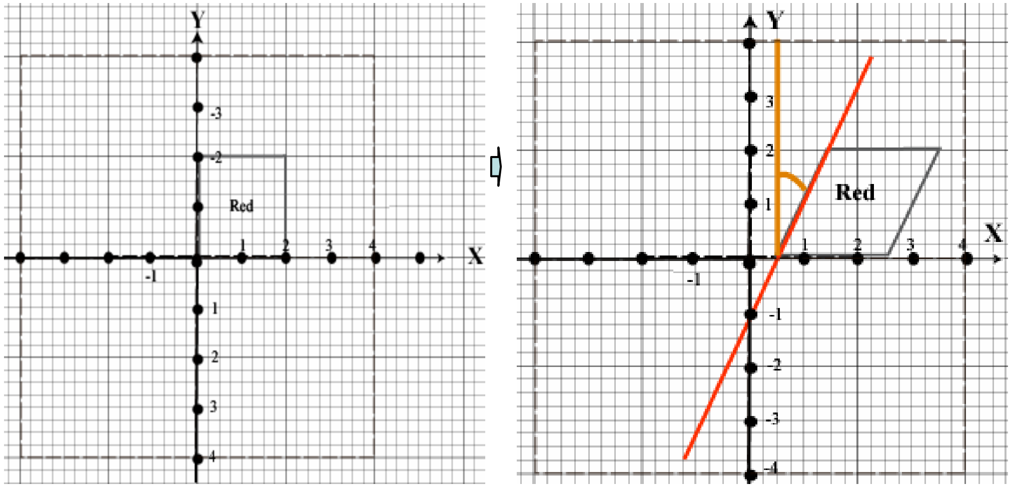
Example:

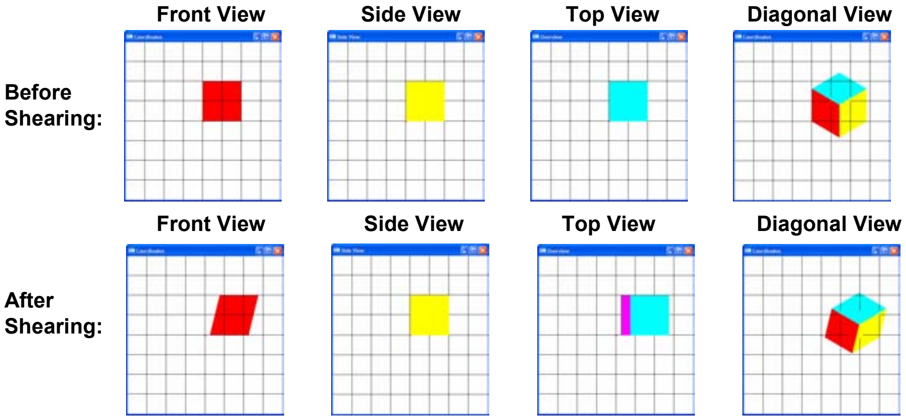
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0, 0, 0)
P = (0,2,0) ➔ p’ = (0, 2, 1.5)

- Case Study: Shxy = 0.25, Yref = -1

Shearing Matrix

$$
\begin{array}{c|cccc|}
 & X & Y & Z & \\
X & 1 & 0.25 & 0 & -0.25*(-1) \\
Y & 0 & 1 & 0 & 0 \\
Z & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 1
\end{array}
$$





XZ direction shearing Matrix with Displacement

$$\text{MxzShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & Shxz & -Shxz*Zref \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

MxzShear will lean the object towards x direction.

Shxz:
    Z increase by 1 unit, X increase by Shxz unit.
Yref:
    Shearing reference point on Z axis

For every P = (X,Y,Z) in the object, after shearing P' = (X',Y',Z')

X' = X
Y' = Y
Z' = Z + Shzx * (Z - Zref)

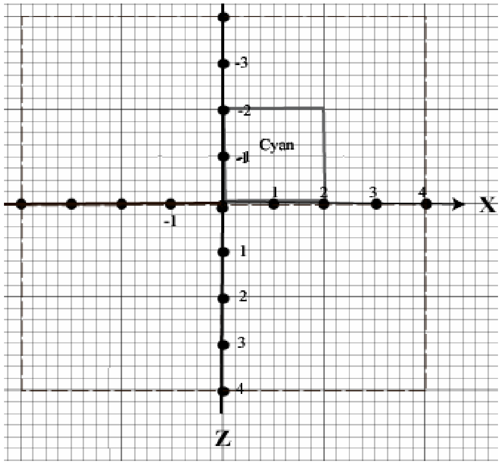
Example:

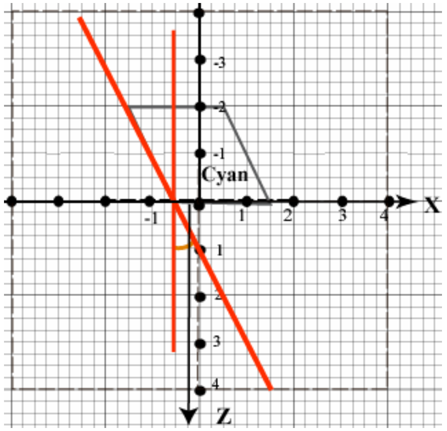
Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P' = (-0.5,0,0)
P = (0,2,0) ➔ P' = (-1.5,2,0)

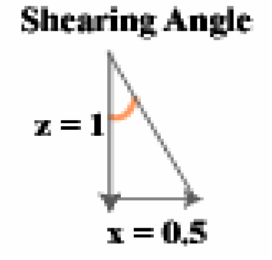
- **Case Study: Shxz = 0.5, Zref = -1**

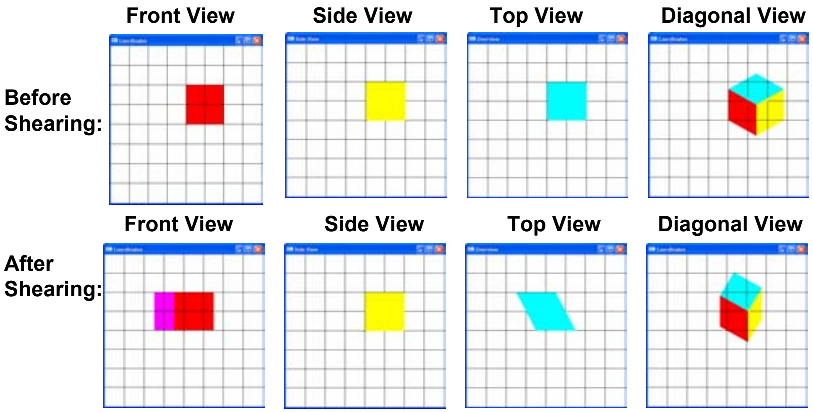
Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0.5 & -0.5*(1) \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$









YX direction shearing Matrix with Displacement

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0.5 & 1 & 0 & -0.5*(-1) \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

$$\text{MyxShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & \text{Shyx} & 1 & 0 & -\text{Shyx*Xref} \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

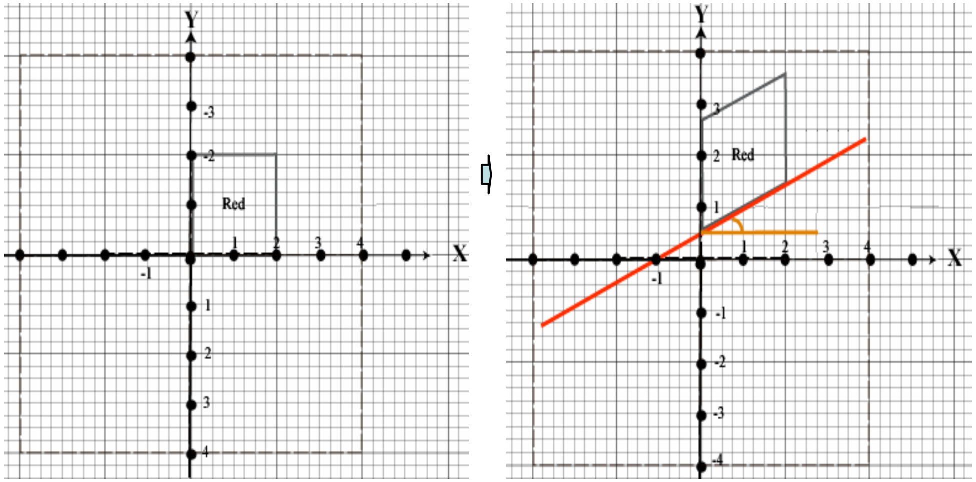
MyxShear will lean the object towards Y direction.

Shyx:
  X increase by 1 unit, Y increase by Shxy unit.
Xref:
  Shearing reference point on X axis

For every P = (X,Y,Z) in the object, after shearing P' = (X',Y',Z')

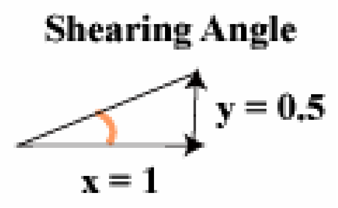
X' = X
Y' = Y + Shyx * (X – Xref)
Z' = Z

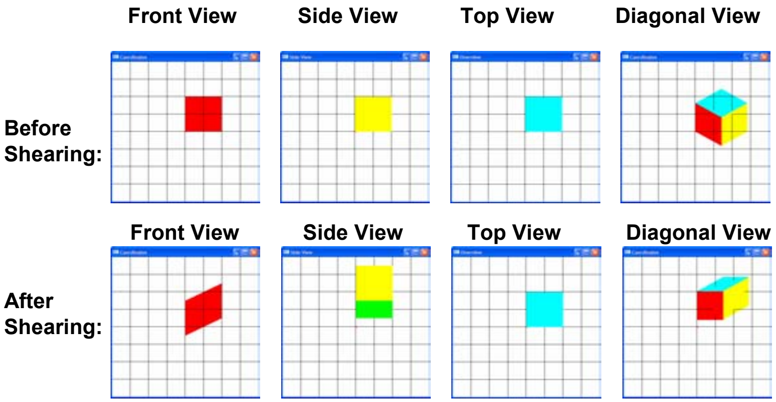
- Example: Shyx = 0.5, Xref = -1



Example:

Shyx = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P' = (0,0.5,0)
P = (2,0,0) ➔ P' = (2,1.5,0)





	Front View	Side View	Top View	Diagonal View
Before Shearing:				
After Shearing:				

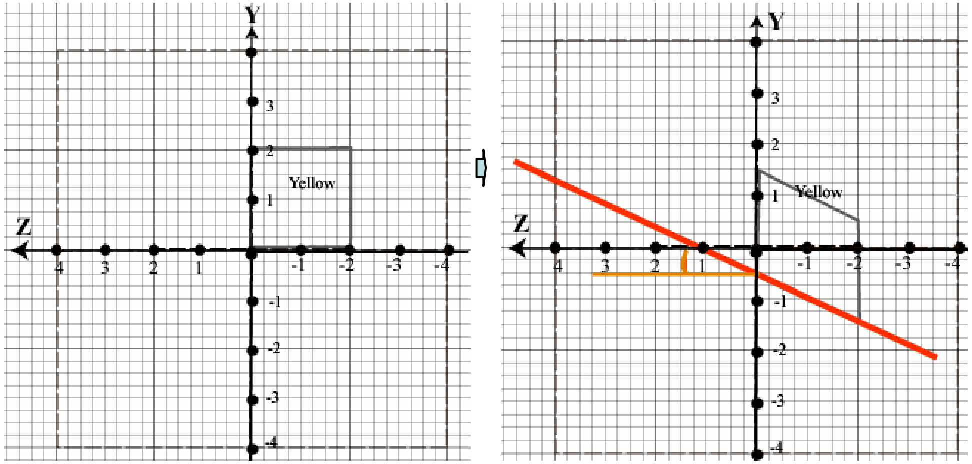
YZ direction shearing Matrix with Displacement

$$\text{MyzShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & \text{Shyz} & -\text{Shyz}^*\text{Zref} \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 1 & 0 \\ Y & 0 & 1 & 0.5 & -0.5^*(1) \\ Z & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

- **Example: Shyz = 0.5, Xref = 1**

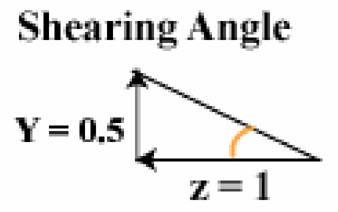


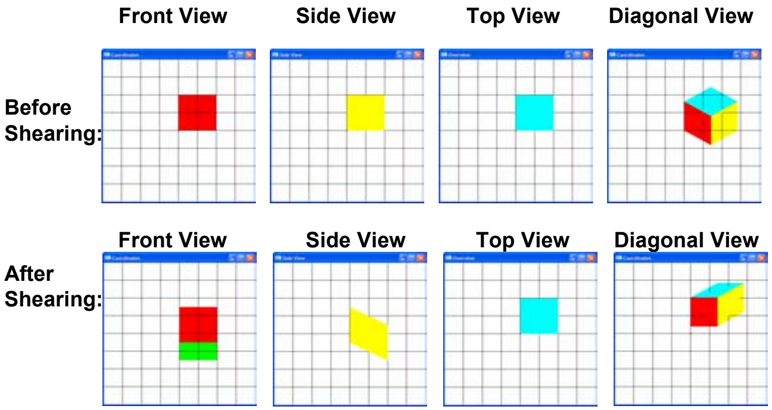
MyxShear will lean the object towards Y direction.

Shyz:
    Z increase by 1 unit, Y increase by Shyz unit.
Zref:
    Shearing reference point on Z axis

For every P = (X,Y,Z) in the object, after shearing P' = (X',Y',Z')

X' = X
Y' = Y + Shyz * (Z – Zref)
Z' = Z





Example:

Shxy = tan(SearingAngle) = 0.5
P = (0,0,0) ➔ P' = (0,-0.5,0)
P = (-2,0,0) ➔ P' = (-2,-1.5,0)

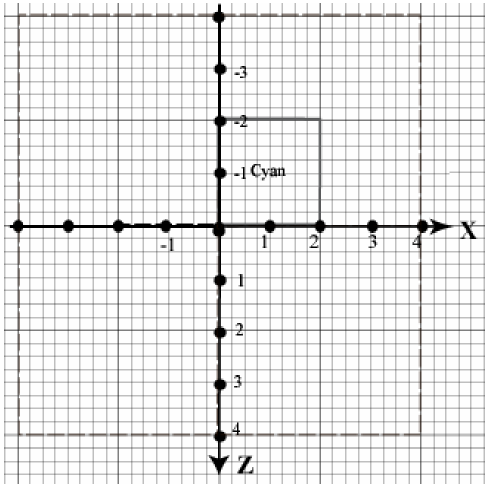
ZX direction shearing Matrix with Displacement

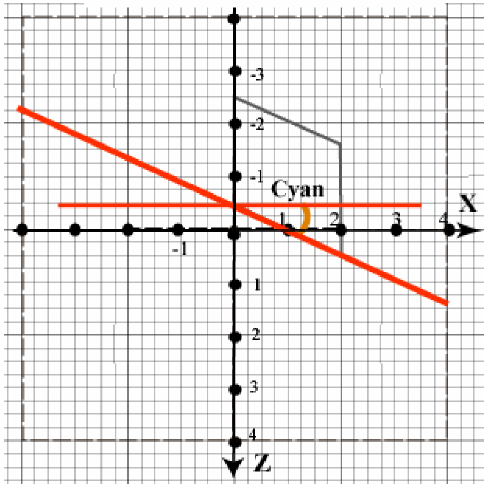
$$\text{MzxShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & \text{Shzx} & 0 & 1 & -\text{Shzx}*\text{Xref} \\ & 0 & 0 & 0 & 1 \end{array}$$

- **Example: Shyz = 0.5, Xref = 1**

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 1 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0.5 & 0 & 1 & -0.5*(1) \\ & 0 & 0 & 0 & 1 \end{array}$$



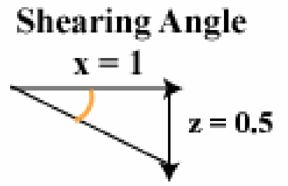


MzxShear will lean the object towards Z direction.

Shzx:
    X increase by 1 unit, Z increase by Shzx unit.
Xref:
    Shearing reference point on X axis

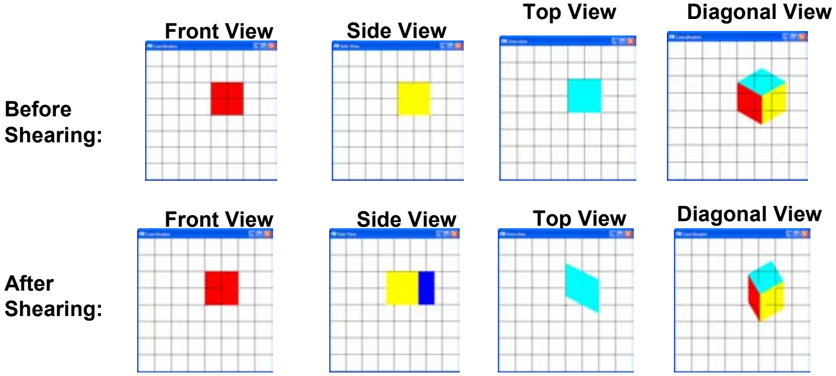
For every P = (X,Y,Z) in the object, after shearing P' = (X',Y',Z')

X' = X
Y' = Y
Z' = Z + Shzx * (X – Xref)



Example:

Shxy = tan(ShearingAngle) = 0.5
P = (0,0,0) ➔ P' = (0, 0, -0.5)
P = (2,0,0) ➔ P' = (2, 0, 0.5)



ZY direction shearing Matrix with Displacement

Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 1 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & 0.5 & 1 & -0.5*(1) \\ & 0 & 0 & 0 & 1 \end{array}$$

$$\text{MzyShear} = \begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ Z & 0 & Shzy & 1 & -Shzy*Yref \\ & 0 & 0 & 0 & 1 \end{array}$$

MzyShear will lean the object towards Z direction.

Shzy:
  Y increase by 1 unit, X increase by Shzy unit.
Yref:
  Shearing reference point on Y axis

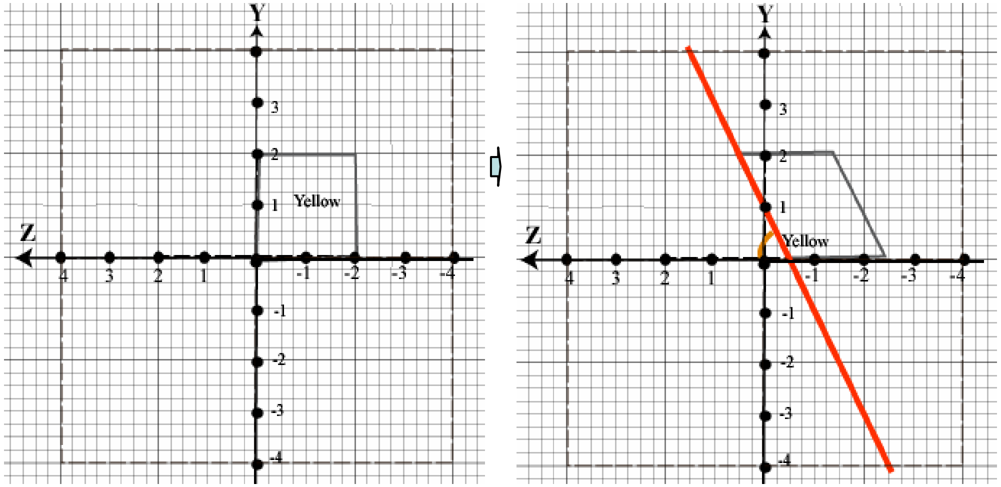
For every P = (X,Y,Z) in the object, after shearing P’ = (X’,Y’,Z’)

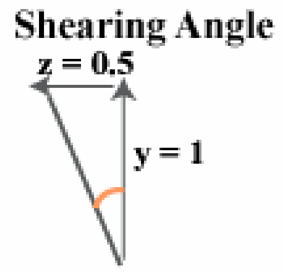
X’ = X
Y’ = Y
Z’ = Z + Shzy * (Y – Yref)

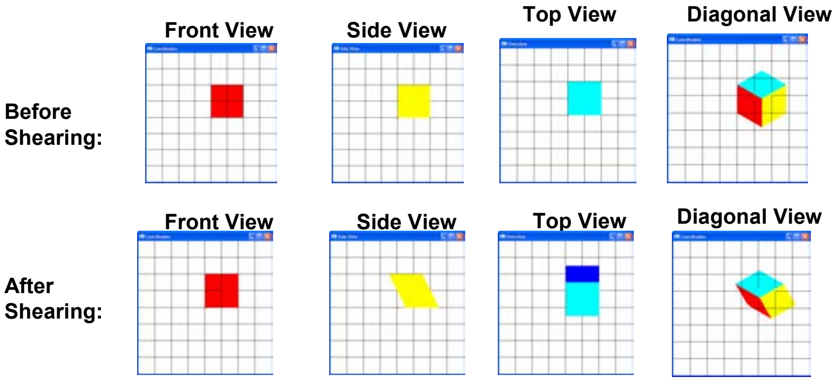
Example:

Shxy = ctn(ShearingAngle) = 0.5
P = (0,0,0) ➔ P’ = (0.5, 0, 0)
P = (2,0,0) ➔ p’ = (2.5, 0, 0)

- Example: Shyz = 0.5, Xref = 1







Affine Transformation

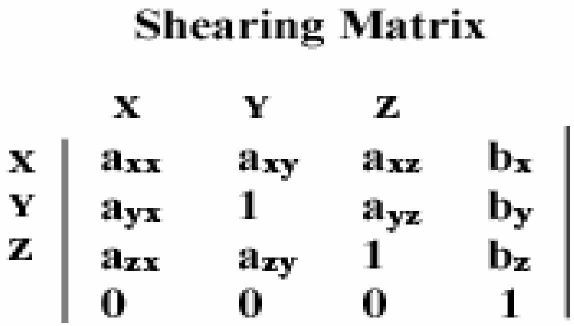
- **Affine Transformation:**

  A coordinate transformation of the form

  $$X' = a_{xx}X + a_{xy}Y + a_{xz}Z + b_x$$
  $$Y' = a_{yx}X + a_{yy}Y + a_{yz}Z + b_x$$
  $$Z' = a_{zx}X + a_{zy}Y + a_{zz}Z + b_x$$

- **Properties:**
  - Parallel lines are transformed into parallel lines
  - Finite points are mapped to finite points, although location might change for those points.

Affine Transformation Matrix:



Shearing Matrix

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & a_{xx} & a_{xy} & a_{xz} & b_x \\ Y & a_{yx} & 1 & a_{yz} & b_y \\ Z & a_{zx} & a_{zy} & 1 & b_z \\ & 0 & 0 & 0 & 1 \end{array}$$

Case Study 3

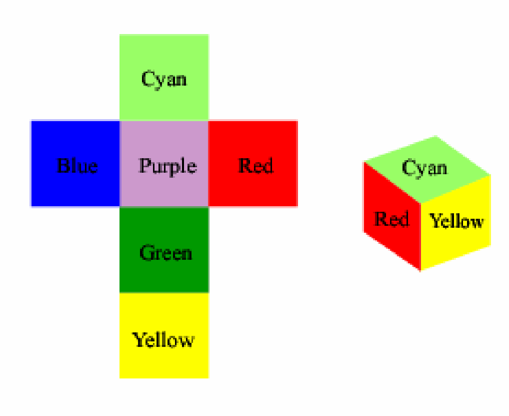
Question: After multiple dimensional shears, are the parallel lines still parallel?

Case Study Setup:

Assume in the world coordinates we have one color cube of size two, whose front is red.

colorcube4 ():

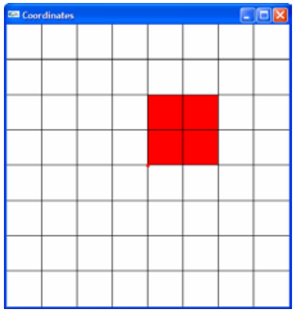
The front left right vertex is at (0,0,0)



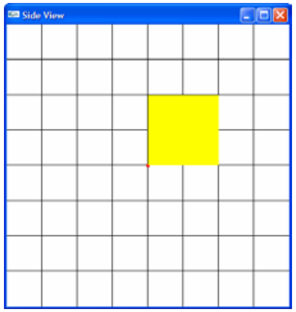
Goal:

We will observe how mutiple shears will transform the cube..

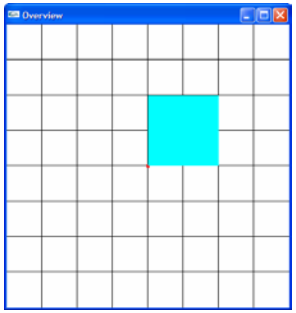
Front View



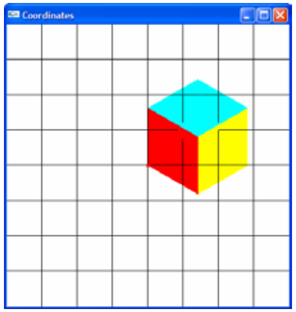
Side View



Top View



Diagonal View



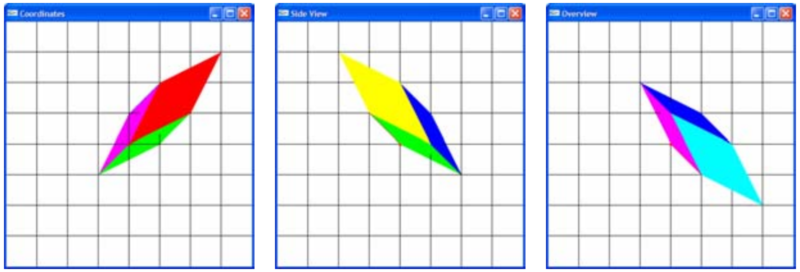
Answer: Yes, shearing is affine transformation, which means parallel lines continue to be parallel lines

Shearing Matrix

- **Example 1:**

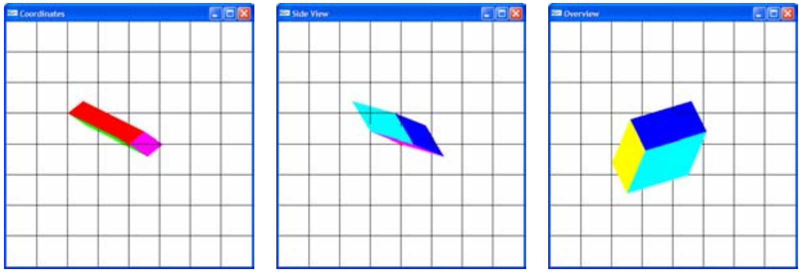
$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0.5 & 0.5 & 0 \\ Y & 0.5 & 1 & 0.5 & 0 \\ Z & 0.5 & 0.5 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

Front View **Side View** **Top View**



- **Example 2:**

$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 1 & 0.25 & 0.3 & 0 \\ Y & 0.5 & 1 & 0.2 & 0 \\ Z & 0.3 & 0.5 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

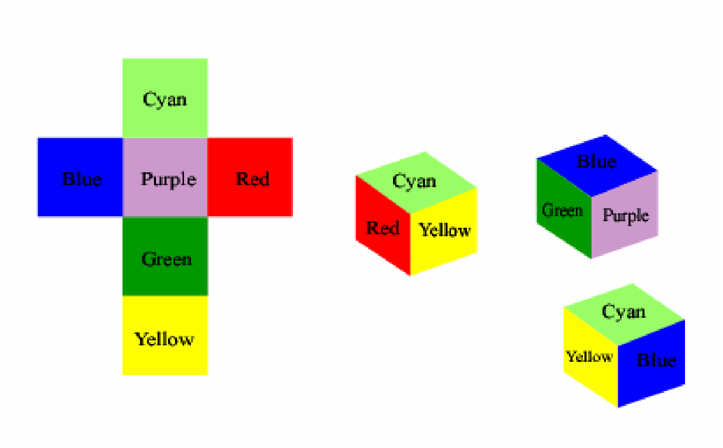


Case Study 4
Shearing of Three Cubes

Case Study Setup:

Assume in the world coordinates we have three cubes of size two. One cube’s front is red, the second one’s front is green, the third one’s front is yellow.

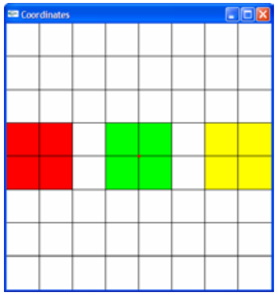
Cube (1): centered at (-3,0,-1)
Cube (2): centered at (3,0,-1)
Cube (3): centered at (0,0,-3)



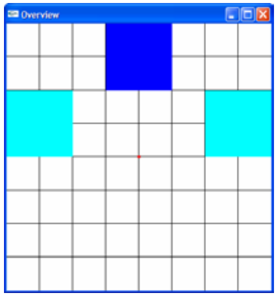
Goal:

We will test the effect of shearing on the view of three colored cubes.

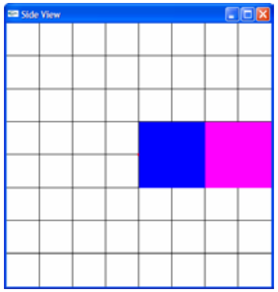
Front View



Top View



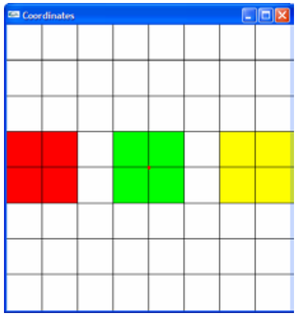
Side View



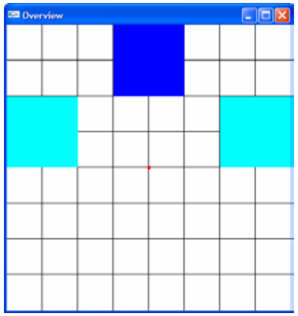
Case Study 6, continued
- Shearing of 3 cubes

Before Shearing

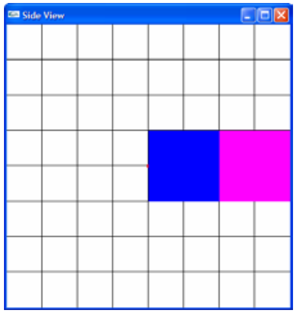
Front View



Top View



Side View

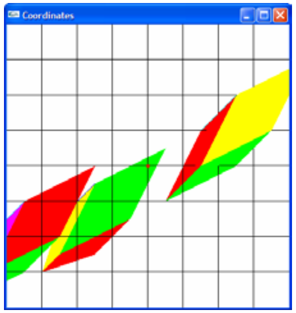


Shearing Matrix

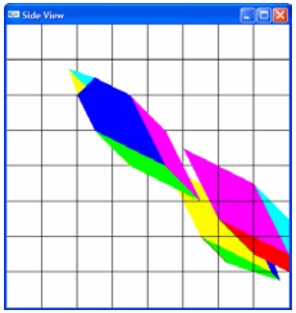
$$\begin{array}{c|cccc|}
 & X & Y & Z & \\
X & 1 & 0.25 & 0.3 & 0 \\
Y & 0.5 & 1 & 0.2 & 0 \\
Z & 0.3 & 0.5 & 1 & 0 \\
 & 0 & 0 & 0 & 1
\end{array}$$

After Shearing

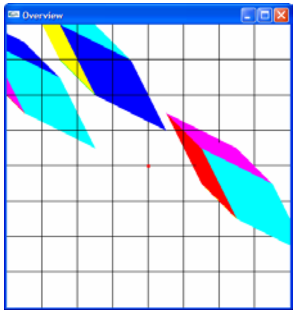
Front View



Top View



Side View

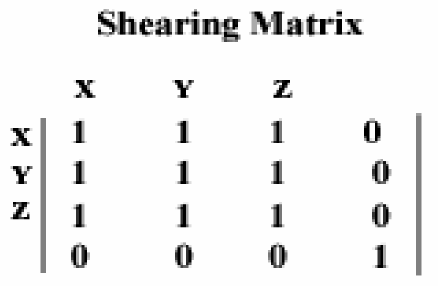


Note:
Color inversion occurs when, due to the limited view space of the camera, those portion of the objects which are outside the viewspace are cut off and their inside color is show instead of their outside color.
(See Top View after shearing)

Case Study 5
What will happen if you use singular matrix for shearing?

- Singular matrix:
  - The matrix whose determinant is 0.
  - Therefore, if the singular matrix is used for transformation, this will not be affine transformation. Because there is no inverse matrix and no way to transform back.

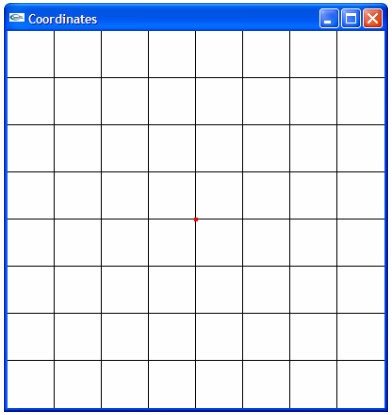
- If you use singular matrix for shearing, you will see only blank screen after the shearing.



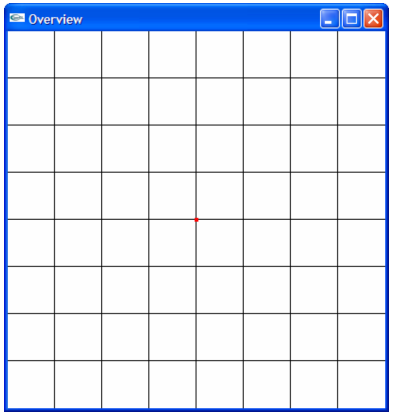
Shearing Matrix

$$\begin{array}{c|cccc|}
 & X & Y & Z & \\
X & 1 & 1 & 1 & 0 \\
Y & 1 & 1 & 1 & 0 \\
Z & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 0 & 1
\end{array}$$

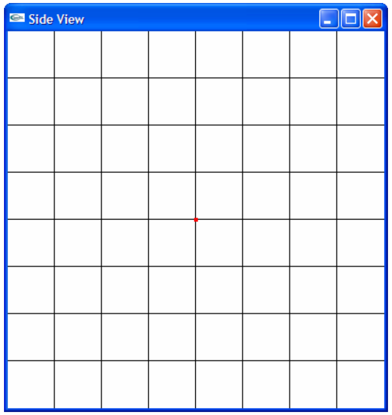
Front View



Side View

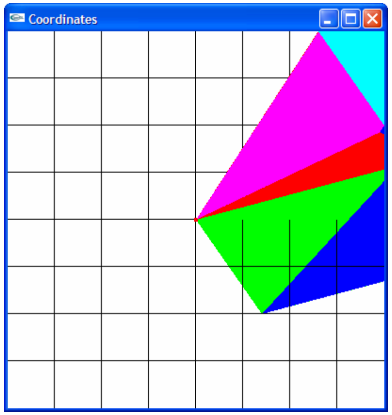


Top View

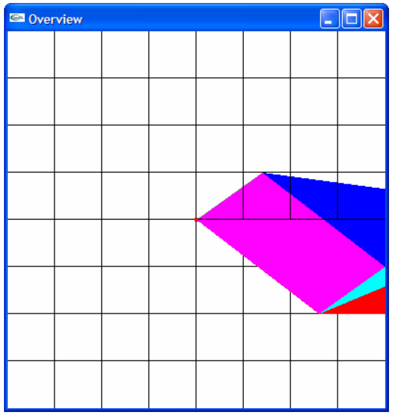


$$\begin{array}{c|cccc|} & X & Y & Z & \\ X & 3 & -1.3 & -0.7 & 0 \\ Y & 0.8 & 2 & 1 & 0 \\ Z & 0.4 & 1 & 0.5 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

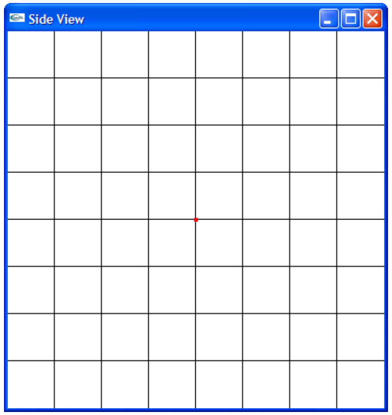
Front View



Side View

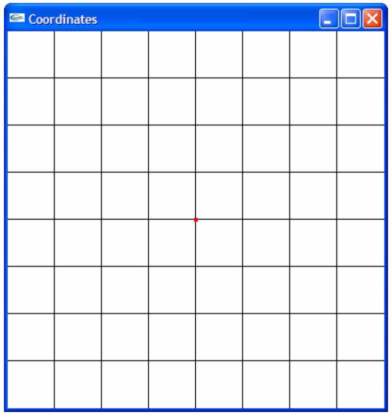


Top View

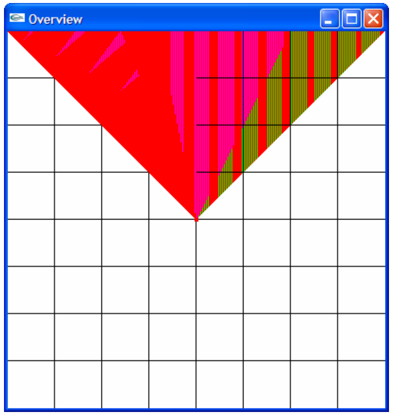


$$\begin{array}{c|cccc|}
 & X & Y & Z & \\
X & 2 & -3 & -1 & 0 \\
Y & -1 & 1.5 & 0.5 & 0 \\
Z & -2 & -3 & 2 & 0 \\
 & 0 & 0 & 0 & 1
\end{array}$$

Front View



Side View



Top View